

The Use of Interleaving for Reducing Radio Loss in Trellis-Coded Modulation Systems

D. Divsalar

Communications Systems Research System

M. K. Simon

Telecommunications Systems Section

In this article, it is demonstrated how the use of interleaving/deinterleaving in trellis-coded modulation (TCM) systems can reduce the SNR loss due to imperfect carrier demodulation references. Both the discrete carrier (phase-locked loop) and suppressed carrier (Costas loop) cases are considered and the differences between the two are clearly demonstrated by numerical results. These results are of great importance for future communication links to the DSN, especially from high Earth orbiters, which may be bandwidth limited.

I. Introduction

In a previous publication [1], the authors demonstrated how in convolutionally coded BPSK systems, the degradation in signal-to-noise ratio (SNR) performance due to imperfect carrier demodulation references (often referred to as *radio* or *noisy reference loss*) could be reduced by employing interleaving/deinterleaving. Specific closed form results were derived for both discrete and suppressed carrier systems and the differences between the two were discussed and numerically illustrated. Here these former results are generalized to trellis-coded MPSK systems [2]. As a numerical example used for illustration, the case of a rate 1/2 trellis-coded QPSK system with a 2-state trellis shall be considered.

II. Upper Bound on the Average Bit Error Probability Performance of TCM

A. Perfect Carrier Phase Synchronization

In previous work by the authors [3-6] on TCM transmitted over a perfectly phase-synchronized additive white Gaussian

noise (AWGN) channel, an upper bound on the average bit error probability was obtained as

$$P_b \leq \sum_{\underline{x}} \sum_{\hat{\underline{x}} \in C} a(\underline{x}, \hat{\underline{x}}) p(\underline{x}) P(\underline{x} \rightarrow \hat{\underline{x}}) \quad (1)$$

where $a(\underline{x}, \hat{\underline{x}})$ is the number of bit errors that occur when the sequence \underline{x} is transmitted and the sequence $\hat{\underline{x}} \neq \underline{x}$ is chosen by the decoder, $p(\underline{x})$ is the a priori probability of transmitting \underline{x} , and C is the set of all coded sequences. Also, in Eq. (1), $P(\underline{x} \rightarrow \hat{\underline{x}})$ represents the pairwise error probability, i.e., the probability that the decoder chooses $\hat{\underline{x}}$ when indeed \underline{x} was transmitted. The upper bound of Eq. (1) is efficiently evaluated using the *transfer function bound* (generating function) approach [7] applied to TCM.

In general, evaluation of the pairwise error probability depends on the proposed decoding metric, the presence or absence of channel state information (CSI), and the type of detection used, i.e., coherent versus differentially coherent.

For the case of interest here, namely, coherent detection with no CSI and a Gaussian metric (optimum for the AWGN channel), it is well known [7] that the pairwise error probability is given by

$$P(\underline{x} \rightarrow \hat{\underline{x}}) \leq \exp \left\{ -\frac{E_s}{4N_0} d^2(\underline{x}, \hat{\underline{x}}) \right\} \quad (2)$$

where

$$d^2(\underline{x}, \hat{\underline{x}}) = \sum_{n \in \eta} |x_n - \hat{x}_n|^2 \triangleq \sum_{n \in \eta} \delta_n^2 \quad (3)$$

represents the sum of the squared Euclidean distances between the two symbol sequences \underline{x} (the correct one) and $\hat{\underline{x}}$ (the incorrect one) and η is the set of all n for which $\hat{x}_n \neq x_n$. Also, in Eq. (3), E_s is the energy per output coded symbol and N_0 is the single-sided noise spectral density.

B. Imperfect Carrier Phase Synchronization

1. **Discrete Carrier (No Interleaving).** When a carrier phase error $\phi(t)$ exists between the received signal and the locally generated demodulation reference, then the result in Eq. (2) is modified as follows.

Assuming the case where the data symbol rate $1/T_s$ is high compared to the loop bandwidth B_L , then $\phi(t)$ can be assumed constant (independent of time), say ϕ , over a number of symbols on the order of $1/B_L T_s$. Since the decoder has no knowledge of ϕ , the decoding metric can make no use of this information and as such is *mismatched* to the channel. Under these conditions, it can be shown (Appendix A) that using the maximum-likelihood metric for a perfectly phase-synchronized AWGN, one obtains

$$P(\underline{x} \rightarrow \hat{\underline{x}} | \phi; \lambda) \leq \begin{cases} \frac{1}{2} \exp \left\{ -\frac{E_s}{4N_0} \sum_{n \in \eta} 4\lambda (\cos \phi + \alpha_n \sin \phi - \lambda) \delta_n^2 \right\}; \\ \sum_{n \in \eta} \delta_n^2 (\cos \phi + \alpha_n \sin \phi) > 0 \\ 1; & \sum_{n \in \eta} \delta_n^2 (\cos \phi + \alpha_n \sin \phi) \leq 0 \end{cases} \quad (4)$$

where

$$\alpha_n \triangleq \sqrt{\frac{4 - |x_n - \hat{x}_n|^2}{|x_n - \hat{x}_n|^2}} = \sqrt{\frac{4 - \delta_n^2}{\delta_n^2}} \quad (5)$$

and $\lambda \geq 0$ is a parameter to be optimized. Note that for $\phi = 0$, the parameter λ can be optimized independent of the summation index n . In particular, the expression $4\lambda(1 - \lambda)$ is maximized by the value $\lambda = 1/2$ which when substituted in Eq. (4) yields Eq. (2) as it should.

Letting $p(\phi)$ denote the probability density function (p.d.f.) of the phase error ϕ , then the average bit error probability is upper bounded by¹

$$P_b \leq \sum_{\underline{x}, \underline{\hat{x}} \in \mathcal{C}} a(\underline{x}, \underline{\hat{x}}) p(\underline{x}) \min_{\lambda} E_{\phi} \{P(\underline{x} \rightarrow \underline{\hat{x}} | \phi; \lambda)\} \quad (6)$$

where $E_{\phi} \{\bullet\}$ denotes statistical averaging over the p.d.f. $p(\phi)$.

To somewhat simplify notation, the Bhattacharyya parameter [7] is introduced

$$D \triangleq \exp \left\{ -\frac{E_s}{4N_0} \right\} \quad (7)$$

in which case

$$\begin{aligned} E_{\phi} \{P(\underline{x} \rightarrow \underline{\hat{x}} | \phi; \lambda)\} &\leq \frac{1}{2} \int_{\mathcal{R}} \left(\prod_{n \in \eta} D^{\delta_n^2} \right)^{[4\lambda(\cos \phi - \lambda)]} \\ &\times \left(\prod_{n \in \eta} D^{\alpha_n \delta_n^2} \right)^{[4\lambda \sin \phi]} p(\phi) d\phi \\ &+ \int_{\bar{\mathcal{R}}} p(\phi) d\phi \end{aligned} \quad (8)$$

where \mathcal{R} is the set of all ϕ in $(-\pi, \pi)$ for which

$$\sum_{n \in \eta} \delta_n^2 (\cos \phi + \alpha_n \sin \phi) > 0 \quad (9)$$

¹ Later on a tighter bound for this case shall be presented by optimizing on λ prior to performing the expectation over ϕ .

and $\bar{\mathcal{R}}$ is the complement of \mathcal{R} , i.e., the remaining values of ϕ in $(-\pi, \pi)$ that do not satisfy Eq. (9). Defining

$$\phi_1 \triangleq \tan^{-1} \frac{\sum_{n \in \eta} \alpha_n \delta_n^2}{\sum_{n \in \eta} \delta_n^2} = \tan^{-1} \frac{\sum_{n \in \eta} \sqrt{\delta_n^2 (4 - \delta_n^2)}}{\sum_{n \in \eta} \delta_n^2} \quad (10)$$

then \mathcal{R} corresponds to the interval $0 \leq |\phi - \phi_1| \leq \pi/2$ and $\bar{\mathcal{R}}$ corresponds to the interval $\pi/2 \leq |\phi - \phi_1| \leq \pi$.

2. Discrete Carrier (With Interleaving). Ordinarily, one thinks of using interleaving/deinterleaving to break up the effects of error bursts in coded communication systems. For example, in TCM systems operating in a multipath fading environment, it has been shown [2] that interleaving/deinterleaving is essential for good performance. To see how it may be applied in systems with noisy carrier phase reference, one can gain an intuitive notion by considering the $\cos \phi$ degradation factor as an “amplitude fade” whose duration is on the order of $1/B_L T_s$ symbols. Thus, if one breaks up this “fade” by interleaving to a depth on the order of $1/B_L T_s$, then, after deinterleaving, the degradation due to $\cos \phi$ and $\sin \phi$ will be essentially *independent* from symbol to symbol. From a mathematical standpoint, this is equivalent to replacing Eq. (4) by

$$P(\underline{x} \rightarrow \hat{\underline{x}} | \underline{\phi}; \lambda) \leq \begin{cases} \frac{1}{2} \exp \left\{ -\frac{E_s}{4N_0} \sum_{n \in \eta} 4\lambda (\cos \phi_n + \alpha_n \sin \phi_n - \lambda) \delta_n^2 \right\}; \\ \sum_{n \in \eta} \delta_n^2 (\cos \phi_n + \alpha_n \sin \phi_n) > 0 \\ 1; & \sum_{n \in \eta} \delta_n^2 (\cos \phi_n + \alpha_n \sin \phi_n) \leq 0 \end{cases} \quad (11)$$

where the ϕ_n 's are independent identically distributed (i.i.d.) random variables with p.d.f. $p(\phi)$ and $\underline{\phi}$ refers to the vector whose components are the ϕ_n 's. The expectation required in

Eq. (6) now involves computation of multidimensional integrals over regions of $\underline{\phi}$ corresponding to the inequalities in Eq. (11). Since, in these regions, the intervals of integration per dimension are *dependent* on one another, the expectation required in Eq. (6) is extremely difficult to compute.

Thus, instead a looser upper bound on conditional pairwise error probability is considered, which has the advantage of not having to separate the multidimensional integration required in Eq. (6) into two disjoint regions. Indeed, it is straightforward to see that the right-hand side of Eq. (11) is upper bounded by the exponential in its first line (without the factor of $1/2$) over the *entire* domain of $\underline{\phi}$, i.e., $\{\phi_n \in (-\pi, \pi); n \in \eta\}$. Hence,

$$P(\underline{x} \rightarrow \hat{\underline{x}} | \underline{\phi}; \lambda) \leq$$

$$\exp \left\{ -\frac{E_s}{4N_0} \sum_{n \in \eta} 4\lambda (\cos \phi_n + \alpha_n \sin \phi_n - \lambda) \delta_n^2 \right\} \\ = \prod_{n \in \eta} \exp \left\{ -\frac{E_s}{N_0} \lambda (\cos \phi_n + \alpha_n \sin \phi_n - \lambda) \delta_n^2 \right\} \quad (12)$$

which is identically equal to the Chernoff bound. Now, substituting Eq. (12) into Eq. (6) gives the much simpler result

$$E_{\phi} \{P(\underline{x} \rightarrow \hat{\underline{x}} | \underline{\phi}; \lambda)\} \leq$$

$$\prod_{n \in \eta} \int_{-\pi}^{\pi} D^{\delta_n^2} [4\lambda (\cos \phi_n + \alpha_n \sin \phi_n - \lambda)] p(\phi_n) d\phi_n \quad (13)$$

3. Suppressed Carrier (No Interleaving). When the carrier synchronization loop used to track the input phase is of the suppressed carrier type (e.g., a Costas loop), then the results of Section II.B.1 have to be somewhat modified since the appropriate domain for ϕ is no longer $(-\pi, \pi)$. In fact, for suppressed carrier tracking of MPSK with a Costas-type loop, and assuming perfect phase ambiguity resolution, ϕ takes on values only in the interval $(-\pi/M, \pi/M)$. Thus, the regions \mathcal{R} and $\bar{\mathcal{R}}$ required in Eq. (8) are reduced relative to those defined below Eq. (10), which assume that ϕ is allowed to take on values in $(-\pi, \pi)$. Specifically, \mathcal{R} will now be the intersection of the intervals $0 \leq |\phi| \leq \pi/M$ and $0 \leq |\phi - \phi_1| \leq \pi/2$ where ϕ_1 is defined in Eq. (10). Similarly, $\bar{\mathcal{R}}$ is defined by the intersection of $\pi/M \leq |\phi| \leq \pi/2$ and $\pi/2 \leq |\phi - \phi_1| \leq \pi$.

It can be shown (see Appendix B) that, for any pair of paths \underline{x} and $\hat{\underline{x}}$ (which define the set η), $\pi/2 - \phi_1 \geq \pi/M$. Thus, the intersection of the intervals $0 \leq |\phi| \leq \pi/M$ and $0 \leq |\phi - \phi_1| \leq \pi/2$ is simply $0 \leq |\phi| \leq \pi/M$, which defines \mathcal{R} , and the intersection of $\pi/M \leq |\phi| \leq \pi/2$ and $\pi/2 \leq |\phi - \phi_1| \leq \pi$ is the null set which defines \mathcal{R} . In short, for suppressed carrier tracking, the second integral in Eq. (8) disappears and the limits on the first integral become $(-\pi/M, \pi/M)$, i.e.,

$$E_\phi \{P(\underline{x} \rightarrow \hat{\underline{x}}|\phi; \lambda)\} \leq \frac{1}{2} \int_{-\pi/M}^{\pi/M} \left(\prod_{n \in \eta} D_n^{\delta_n^2} \right)^{[4\lambda(\cos \phi - \lambda)]} \times \left(\prod_{n \in \eta} D_n^{\alpha_n \delta_n^2} \right)^{[4\lambda \sin \phi]} p(\phi) d\phi \quad (14)$$

The significance of the second integral in Eq. (8) being equal to zero will be mentioned shortly relative to a discussion of *irreducible error probability*.

4. Suppressed Carrier (With Interleaving). Once again, assuming suppressed carrier tracking of MPSK with a Costas-type loop, and perfect phase ambiguity resolution, one obtains, analogous to Eq. (13),²

$$E_\phi \{P(\underline{x} \rightarrow \hat{\underline{x}}|\phi; \lambda)\} \leq \frac{1}{2} \prod_{n \in \eta} \left\{ \int_{-\pi/M}^{\pi/M} D_n^{\delta_n^2 [4\lambda(\cos \phi_n + \alpha_n \sin \phi_n - \lambda)]} p(\phi_n) d\phi_n \right\} \quad (15)$$

III. Carrier Synchronization Loop Statistical Model and Average Pairwise Error Probability Evaluation

To evaluate Eq. (6) using Eqs. (8), (13), (14), or (15), one must specify the functional form of the p.d.f. $p(\phi)$ of the

modulo- 2π -reduced phase error ϕ . For a discrete carrier synchronization loop of the phase-locked type, $p(\phi)$ is given by the Tikhonov p.d.f. [8]

$$p(\phi) = \begin{cases} \frac{\exp(\rho \cos \phi)}{2\pi I_0(\rho)}; & |\phi| \leq \pi \\ 0; & \text{otherwise} \end{cases} \quad (16)$$

where ρ is the SNR in the loop bandwidth.

In order to allow evaluation of Eq. (6) in closed form, one must recognize that, for the case of no interleaving, Eq. (8) can be further upper bounded by using $(-\pi, \pi)$ instead of \mathcal{R} in the first integral. Then, making this replacement one obtains

$$\min_{\lambda} E_\phi \{P(\underline{x} \rightarrow \hat{\underline{x}}|\phi; \lambda)\} \leq \min_{\lambda} \left(\frac{1}{2} \exp \left\{ d^2(\underline{x}, \hat{\underline{x}}) \lambda^2 \frac{E_s}{N_0} \left\{ \frac{I_0(\rho')}{I_0(\rho)} + I \right\} \right\} \right) \quad (17)$$

where

$$\rho' \triangleq \left\{ \left(\rho - d^2(\underline{x}, \hat{\underline{x}}) \lambda \frac{E_s}{N_0} \right)^2 + \left[d_1^2 \lambda \frac{E_s}{N_0} \right]^2 \right\}^{1/2}$$

$$d_1^2 \triangleq \sum_{n \in \eta} \alpha_n \delta_n^2$$

$$I = \frac{1}{2\pi I_0(\rho)} \left[\int_{\pi/2+\phi_1}^{\pi} \exp(\rho \cos \phi) d\phi + \int_{-\pi}^{-\pi/2+\phi_1} \exp(\rho \cos \phi) d\phi \right]$$

When Eq. (17) is substituted into Eq. (6), the term I will contribute an *irreducible error probability*, i.e., the system will exhibit a finite error probability when ρ is held fixed and E_s/N_0 approaches infinity.

²Note that the factor of 1/2 can be included here since for $0 \leq |\phi_n| \leq \pi/M$; $n \in \eta$, the condition on the first line of Eq. (11) is always satisfied (see Appendix B) and thus one needs not use the looser upper bound of Eq. (12).

When interleaving is employed, Eq. (15) (minimized over λ) together with Eq. (16) becomes

$$\begin{aligned} \min_{\lambda} E_{\phi} \{P(\underline{x} \rightarrow \hat{\underline{x}} | \phi; \lambda)\} \\ \leq \min_{\lambda} \left\{ \prod_{n \in \eta} \exp \left\{ \delta_n^2 \lambda^2 \frac{E_s}{N_0} \right\} \frac{I_0(\rho_n)}{I_0(\rho)} \right\} \\ = \min_{\lambda} \left\{ \exp \left\{ d^2(\underline{x}, \hat{\underline{x}}) \lambda^2 \frac{E_s}{N_0} \right\} \prod_{n \in \eta} \frac{I_0(\rho_n)}{I_0(\rho)} \right\} \end{aligned} \quad (18)$$

$$\rho_n \triangleq \left(\left(\rho - \delta_n^2 \lambda \frac{E_s}{N_0} \right)^2 + \left(\alpha_n \delta_n^2 \lambda \frac{E_s}{N_0} \right)^2 \right)^{1/2}$$

For suppressed carrier tracking with an M-phase Costas loop, $p(\phi)$ again has a Tikhonov-type p.d.f. which is given by

$$p(\phi) = \begin{cases} \frac{\exp(\rho \cos M\phi)}{(2\pi/M)I_0(\rho)} & ; \quad |\phi| \leq \frac{\pi}{M} \\ 0; & \text{otherwise} \end{cases} \quad (19)$$

Here ρ is the "effective" loop SNR which includes the effects of signal \times signal ($S \times S$), signal \times noise ($S \times N$), and noise \times noise ($N \times N$) degradations commonly referred to as "squaring loss" or, more accurately " M th power loss" [8]. Since suppressed carrier systems of this type derive their carrier demodulation reference from the data-bearing signal, the loop SNR, ρ , is directly proportional to E_s/N_0 ; thus there can be no irreducible error probability since $\rho \rightarrow \infty$ when $E_s/N_0 \rightarrow \infty$. Furthermore, for perfect phase ambiguity resolution, we have previously shown that, for no interleaving, the term I is identically zero since the region $\bar{\mathcal{R}}$ corresponds to the null set. Thus, the average pairwise error probability results become

$$\begin{aligned} \min_{\lambda} E_{\phi} \{P(\underline{x} \rightarrow \hat{\underline{x}} | \phi; \lambda)\} \\ \leq \min_{\lambda} \left\{ \frac{1}{2} \exp \left\{ d^2(\underline{x}, \hat{\underline{x}}) \lambda^2 \frac{E_s}{N_0} \right\} \frac{f(\rho)}{I_0(\rho)} \right\} \end{aligned} \quad (20)$$

$$\begin{aligned} f(\rho) \triangleq \frac{M}{2\pi} \int_{-\pi/M}^{\pi/M} \exp \left\{ \rho \cos M\phi - \lambda \frac{E_s}{N_0} \right. \\ \left. \times (d^2(\underline{x}, \hat{\underline{x}}) \cos \phi + d_1^2 \sin \phi) \right\} d\phi \end{aligned}$$

for no interleaving and

$$\min_{\lambda} E_{\phi} \{P(\underline{x} \rightarrow \hat{\underline{x}} | \phi; \lambda)\}$$

$$\begin{aligned} \leq \min_{\lambda} \left\{ \prod_{n \in \eta} \exp \left\{ \delta_n^2 \lambda^2 \frac{E_s}{N_0} \right\} \frac{f_n(\rho)}{I_0(\rho)} \right\} \\ = \min_{\lambda} \left\{ \exp \left\{ d^2(\underline{x}, \hat{\underline{x}}) \lambda^2 \frac{E_s}{N_0} \right\} \prod_{n \in \eta} \frac{f_n(\rho)}{I_0(\rho)} \right\} \end{aligned}$$

$$f_n(\rho) \triangleq \frac{M}{2\pi} \int_{-\pi/M}^{\pi/M} \exp \left\{ \rho \cos M\phi \right.$$

$$\left. - \lambda \frac{E_s \delta_n^2}{N_0} (\cos \phi + \alpha_n \sin \phi) \right\} d\phi$$

(21)

for the case of interleaving.

In arriving at Eqs. (17), (18), (20), and (21), the "same type" of Chernoff bound has been assumed, in the sense that in all cases, the minimization over λ was performed *after* the averaging over ϕ . The principal reason for doing this is to allow comparison of performance with and without interleaving using bounds with "similar degrees of looseness." For the case of no interleaving, one can actually achieve a tighter bound than that given above by performing the minimization over λ on the conditional pairwise probability in Eq. (4). When this is done, one obtains

$$\lambda_{\text{opt}} = \frac{1}{2} [\cos \phi + \zeta(\underline{x}, \hat{\underline{x}}) \sin \phi]$$

$$\zeta(\underline{x}, \hat{\underline{x}}) \triangleq \frac{d_1^2}{d^2(\underline{x}, \hat{\underline{x}})} = \frac{\sum_{n \in \eta} \sqrt{\delta_n^2 (4 - \delta_n^2)}}{\sum_{n \in \eta} \delta_n^2}$$

(22)

and Eq. (4) becomes

$$P(\underline{x} \rightarrow \hat{\underline{x}} | \phi) \leq \begin{cases} \frac{1}{2} \exp \left\{ -\frac{E_s}{4N_0} d^2(\underline{x}, \hat{\underline{x}}) \right. \\ \quad \times (\cos \phi + \zeta(\underline{x}, \hat{\underline{x}}) \sin \phi)^2 \Big\}; \\ \sum_{n \in \eta} \delta_n^2 (\cos \phi + \alpha_n \sin \phi) > 0 \\ 1; \quad \sum_{n \in \eta} \delta_n^2 (\cos \phi + \alpha_n \sin \phi) \leq 0 \end{cases} \quad (23)$$

where $d^2(\underline{x}, \hat{\underline{x}})$ is again defined by Eq. (3).³

Unfortunately, the integral of Eq. (23) over the p.d.f.s of Eqs. (16) and (19) cannot be obtained in closed form. Defining the integral

$$L(\beta; J) = \int_{\mathcal{R}} \exp \left\{ -\frac{E_s d^2(\underline{x}, \hat{\underline{x}})}{4N_0} [\cos \phi + \beta \sin \phi]^2 \right\} \\ \times \frac{\exp(\rho \cos J\phi)}{\left(\frac{2\pi}{J}\right) I_0(\rho)} d\phi \quad (24)$$

The average pairwise error probabilities are now as follows:

Discrete Carrier

$$E_\phi \left\{ \min_{\lambda} P(\underline{x} \rightarrow \hat{\underline{x}} | \phi; \lambda) \right\} \leq \frac{1}{2} L(\zeta(\underline{x}, \hat{\underline{x}}); 1) + I \quad (25)$$

where I is defined in Eq. (17).

Suppressed Carrier

$$E_\phi \left\{ \min_{\lambda} P(\underline{x} \rightarrow \hat{\underline{x}} | \phi; \lambda) \right\} \leq \frac{1}{2} L(\zeta(\underline{x}, \hat{\underline{x}}); M) \quad (26)$$

where the region \mathcal{R} in the integral of Eq. (24) now corresponds to the interval $(-\pi/M, \pi/M)$.

Using Eqs. (25) and (26) (rather than Eqs. 17 and 20) will result in a smaller improvement in performance due to interleaving/deinterleaving since Eqs. (25) and (26) result in a tighter bound on P_b (no interleaving).

IV. A Trellis-Code Example

Consider a rate 1/2 trellis-coded QPSK using a simple 2-state trellis. The code trellis structure with the appropriate QPSK symbol assignment is illustrated in Fig. 1 and the corresponding pair-state transition diagram is shown in Fig. 2 where a, b, and c are branch label gains to be specified below. The transfer function of the pair-state diagram is

$$T(D, I) = \frac{Iac}{1 - Ib} \quad (27)$$

A. No Interleaving

For the case of no interleaving, one has

$$\left. \frac{dT(D, I)}{dI} \right|_{I=1} = \frac{ac}{(1-b)^2} = \sum_{k=0}^{\infty} (k+1) acb^k \quad (28)$$

where

$$a = D^{16\lambda(\cos \phi - \lambda)} \\ b = c = D^{8\lambda(\cos \phi + \sin \phi - \lambda)} \quad (29)$$

and D is the Bhattacharyya parameter for the ideal AWGN channel, namely

$$D = \exp \left\{ -\frac{E_s}{4N_0} \right\} \quad (30)$$

Using Eq. (28) and the result of Eq. (23), the upper bound on average bit error probability can be represented as

$$P_b \leq \sum_{k=0}^{\infty} (k+1) \int P_k(\phi) p(\phi) d\phi \quad (31)$$

³Note that Eq. (23) can be obtained directly by applying the bound of Eq. (A-15) to Eq. (A-7) together with Eq. (A-11).

where

$$P_k(\phi) = \begin{cases} \frac{1}{2} D^{2(k+3)} \left(\cos \phi + \frac{k+1}{k+3} \sin \phi \right)^2; \\ \phi - \tan^{-1} \frac{k+1}{k+3} < \frac{\pi}{2} \\ 1; \quad \phi - \tan^{-1} \frac{k+1}{k+3} \geq \frac{\pi}{2} \end{cases} \quad (32)$$

For discrete carrier synchronization, $p(\phi)$ is given by Eq. (16) and for suppressed carrier tracking with a 4-phase Costas loop, $p(\phi)$ is given by Eq. (19) with $M = 4$.

B. Interleaving

When interleaving is employed, then analogous to Eq. (28) one has

$$\left. \frac{dT(D, I)}{dI} \right|_{I=1} = \frac{\bar{a}\bar{c}}{(1-\bar{b})^2} = \sum_{k=0}^{\infty} (k+1) \bar{a} \bar{c} \bar{b}^k \quad (33)$$

where

$$\begin{aligned} \bar{a} &= \int D^{16\lambda(\cos \phi - \lambda)} p(\phi) d\phi \\ \bar{b} = \bar{c} &= \int D^{8\lambda(\cos \phi + \sin \phi - \lambda)} p(\phi) d\phi \end{aligned} \quad (34)$$

For discrete carrier synchronization, Eq. (34) can be represented in closed form as

$$\bar{a} = \exp \left\{ \frac{4\lambda^2 E_s}{N_0} \right\} \frac{I_0(\rho_0)}{I_0(\rho)} \quad (35)$$

$$\bar{b} = \exp \left\{ \frac{2\lambda^2 E_s}{N_0} \right\} \frac{I_0 \left(\sqrt{\left(\rho - \frac{2\lambda E_s}{N_0} \right)^2 + \left(\frac{2\lambda E_s}{N_0} \right)^2} \right)}{I_0(\rho)}$$

where ρ_0 is defined by

$$\rho_0 \triangleq \left| \rho - \frac{4\lambda E_s}{N_0} \right| \quad (36)$$

Using Eq. (33), an expression for the upper bound on average bit error probability, analogous to Eq. (31), is given by

$$P_b \leq \sum_{k=0}^{\infty} (k+1) \min_{\lambda} \gamma \bar{a}(\bar{b})^{k+1} \quad (37)$$

where $\gamma = 1$ for discrete carrier and $\gamma = 1/2$ for suppressed carrier.

The upper bounds of Eqs. (31) and (37) are plotted in Figs. 3 and 4 for discrete carrier tracking and loop SNRs $\rho = 13$ and 15 dB, respectively. In Figs. 5 and 6, the comparable results for the suppressed carrier case are illustrated. Here a 4-phase Costas loop with integrate-and-dump arm filters has been assumed whose equivalent loop SNR is

$$\rho = \frac{1}{16} \left(\frac{S}{N_0 B_L} \right) \mathcal{S}_L = \frac{1}{16} \left(\frac{E_b}{N_0} \right) \left(\frac{1}{B_L T_b} \right) \mathcal{S}_L \quad (38)$$

with \mathcal{S}_L now denoting the “4th power loss” and given by [8]

$$\mathcal{S}_L = \left[1 + \frac{9}{2} \left(\frac{E_s}{N_0} \right)^{-1} + 6 \left(\frac{E_s}{N_0} \right)^{-2} + \frac{3}{2} \left(\frac{E_s}{N_0} \right)^{-3} \right]^{-1} \quad (39)$$

Also, in evaluating the numerical results, the series in Eqs. (31) and (37) has been truncated to 15 terms.

V. Concluding Remarks

It has been shown that by interleaving the transmitted coded symbols in a trellis-coded system, the radio loss can be significantly reduced. The amount of this reduction depends on the particular trellis code used and the region of operation of the system as characterized by such parameters as bit error rate and loop SNR. In this article, a simple example (2-state, rate 1/2 trellis-coded QPSK) has been used strictly for the purpose of illustrating the theoretical results. More complex trellis codes with a larger number of modulation levels and a larger number of states will show even more gain due to interleaving.

In general, whether or not coding and interleaving are employed, suppressed carrier systems have smaller radio losses than discrete carrier systems since they are not subject to

irreducible error probability. This is true despite the fact that for practical passive arm filters (e.g., RC filters) in the suppressed carrier tracking loop (Costas loop), one will experience larger squaring losses and thus larger radio losses than those shown here for active integrate-and-dump arm filters [9]. Thus, if the radio loss is, without interleaving, small (as tends

to be true in suppressed carrier systems), the use of interleaving cannot be of much additional help. Nevertheless, if the system can tolerate the delay associated with the interleaving/deinterleaving process, it is useful to include it in the system design since it also helps to reduce other impairments of a bursty nature such as intersymbol interference, fading, etc.

References

- [1] D. Divsalar, M. K. Simon, and J. H. Yuen, "The Use of Interleaving for Reducing Radio Loss in Convolutionally Coded Systems," *TDA Progress Report 42-96*, vol. October–December 1988, Jet Propulsion Laboratory, Pasadena, California, pp. 21–39, February 15, 1989.
- [2] G. Ungerboeck, "Trellis Coded Modulation with Redundant Signal Sets, Parts I and II," *IEEE Communications Magazine*, vol. 25, no. 2, pp. 5–21, February 1987.
- [3] M. K. Simon and D. Divsalar, *Combined Trellis Coding with Asymmetric MPSK Modulation*, JPL Publication 85-24 (MSAT-X Report No. 109), May 1, 1985. Also see *IEEE Trans. Comm.*, vol. COM-35, no. 2, pp. 130–141, February 1987.
- [4] D. Divsalar and M. K. Simon, *Trellis Coded Modulation for 4800-9600 bps Transmission over a Fading Mobile Satellite Channel*, JPL Publication 86-8 (MSAT-X Report No. 129), June 1, 1986. Also in *IEEE Journal on Selected Areas in Communications, Special Issue on Fading Channels*, vol. SAC-5, no. 2, pp. 162–175, February 1987.
- [5] D. Divsalar and M. K. Simon, *Multiple Trellis Coded Modulation (MTCM)*, JPL Publication 86-44 (MSAT-X Report No. 141), November 15, 1986. Also in *IEEE Trans. Comm.*, vol. 36, no. 4, pp. 410–419, April 1988.
- [6] D. Divsalar and M. K. Simon, "The Design of Trellis Coded MPSK for Fading Channels: Performance Criteria, *IEEE Trans. Comm.*, vol. 36, no. 9, pp. 1004–1012, September 1988.
- [7] A. J. Viterbi and J. K. Omura, *Principles of Digital Communication and Coding*, New York, NY: McGraw-Hill, Inc., 1979.
- [8] W. C. Lindsey and M. K. Simon, *Telecommunications Systems Engineering*, Englewood Cliffs, NJ: Prentice-Hall, Inc., 1978.
- [9] W. C. Lindsey and M. K. Simon, "Optimum Performance of Suppressed Carrier Receivers with Costas Loop Tracking," *IEEE Trans. Comm.*, vol. COM-25, no. 2, pp. 215–227, February 1977.

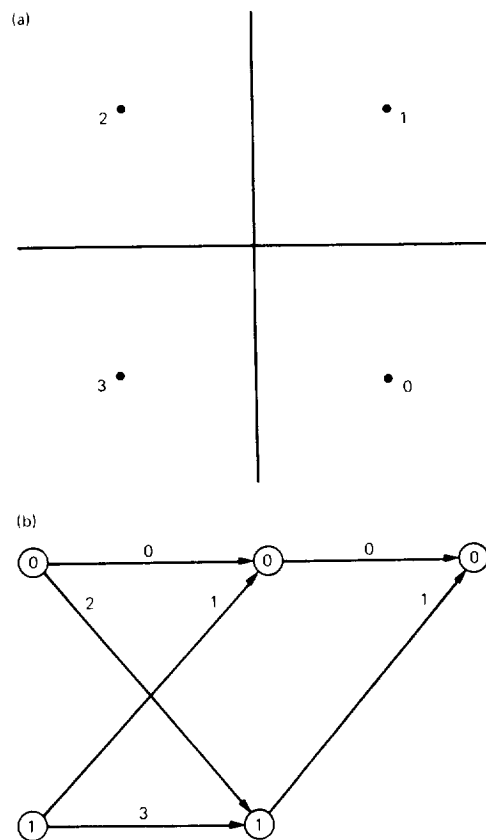


Fig. 1. (a) QPSK signal point constellation and (b) trellis diagram showing QPSK signal assignments to branches.

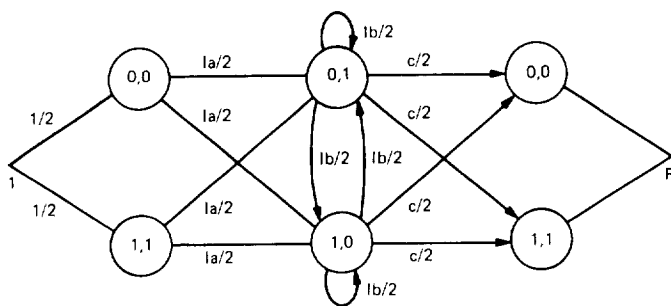


Fig. 2. Pair-state transition diagram for trellis diagram of Fig. 1.

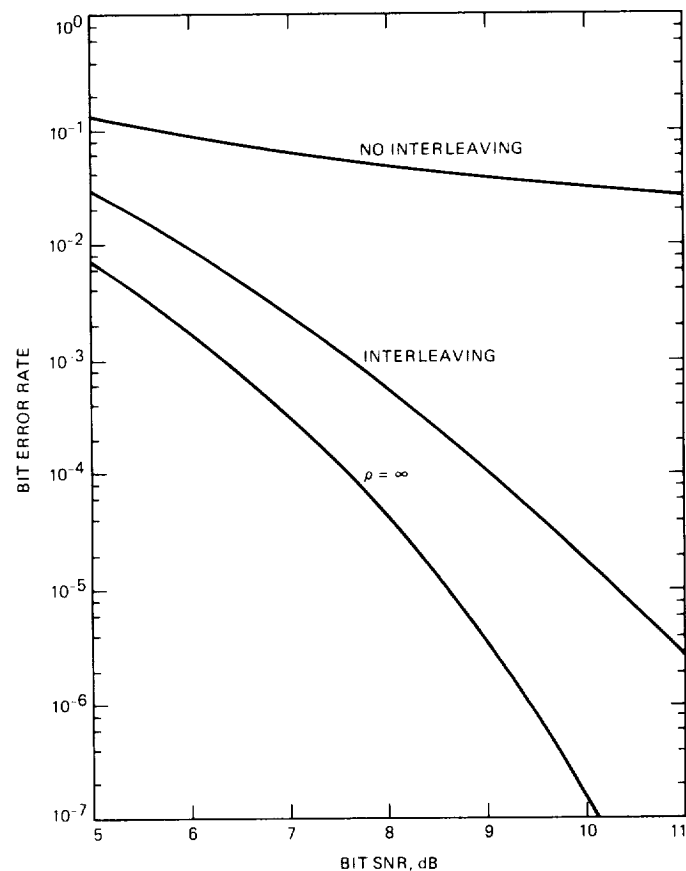


Fig. 3. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/2, trellis-coded QPSK; 2 states; loop SNR = 13 dB; discrete carrier.

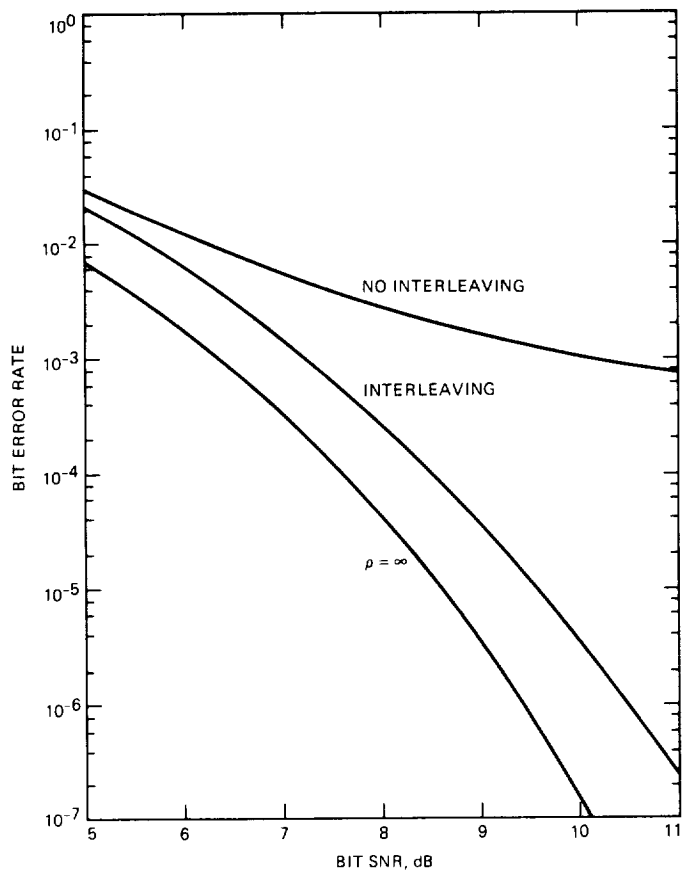


Fig. 4. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/2, trellis-coded QPSK; 2 states; loop SNR = 15 dB; discrete carrier.

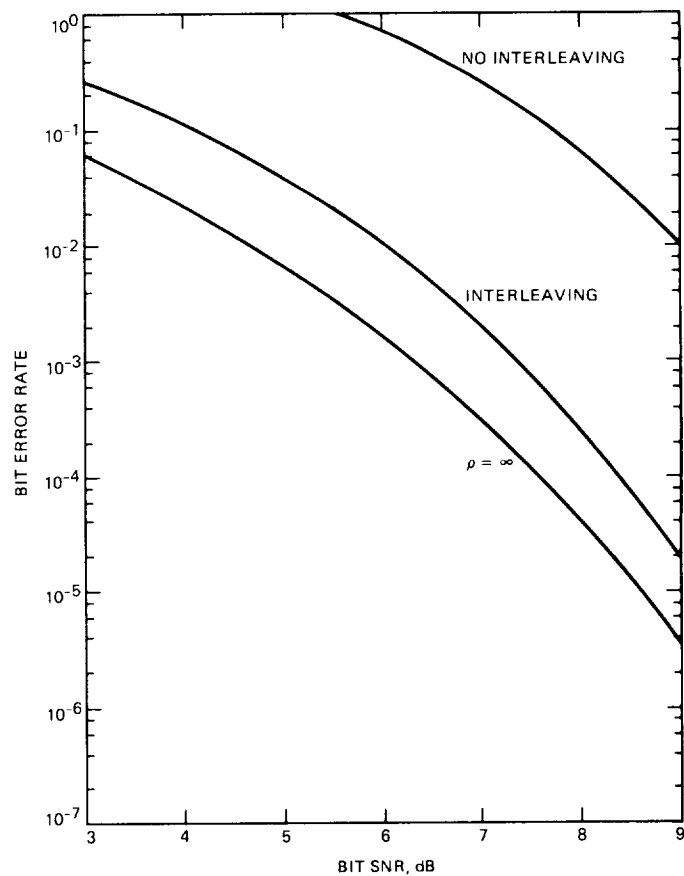


Fig. 5. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/2, trellis-coded QPSK; 2 states; $1/B_L T_b = 10$; suppressed carrier.

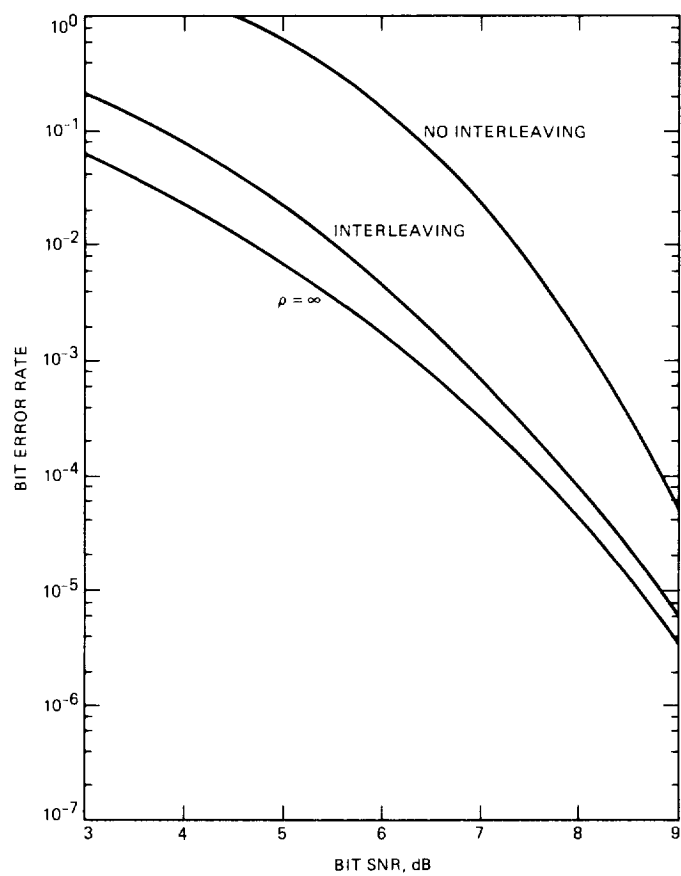


Fig. 6. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/2, trellis-coded QPSK; 2 states; $1/B_L T_b = 20$; suppressed carrier.

Appendix A

Derivation of an Upper Bound on the Pairwise Error Probability for Trellis Coded MPSK with Imperfect Carrier Phase Reference

Let $\underline{y} = (y_1, y_2, \dots, y_N)$ denote the received sequence when the normalized (to unit power) sequence of MPSK symbols $\underline{x} = (x_1, x_2, \dots, x_N)$ is transmitted. A *pairwise error* occurs if $\hat{\underline{x}} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N) \neq \underline{x}$ is chosen by the receiver, which, if the receiver uses a distance metric to make this decision, implies \underline{y} is closer to $\hat{\underline{x}}$ than to \underline{x} . Assuming that distance metric which is maximum-likelihood for ideal coherent detection (perfect carrier phase reference), then such an error occurs whenever

$$\sum_{n=1}^N |y_n - \hat{x}_n|^2 < \sum_{n=1}^N |y_n - x_n|^2 \quad (\text{A-1})$$

Since MPSK is a constant envelope signaling set, one has $|x_n|^2 = |\hat{x}_n|^2 = a$ a constant, and Eq. (A-1) reduces to

$$\sum_{n=1}^N \text{Re} \{ y_n \hat{x}_n^* \} > \sum_{n=1}^N \text{Re} \{ y_n x_n^* \} \quad (\text{A-2})$$

Letting n_n represent the additive noise in the n th signaling interval, and ϕ_n the phase shift introduced by imperfect carrier demodulation in that same interval, then y_n and x_n are related by

$$y_n = x_n e^{j\phi_n} + n_n; \quad n = 1, 2, \dots, N \quad (\text{A-3})$$

Substituting Eq. (A-3) into Eq. (A-2) and simplifying gives

$$\text{Re} \left\{ \sum_{n \in \eta} (\hat{x}_n - x_n)^* n_n \right\} > \text{Re} \left\{ \sum_{n \in \eta} x_n (x_n - \hat{x}_n)^* e^{j\phi_n} \right\} \quad (\text{A-4})$$

where η is the set of all n such that $x_n \neq \hat{x}_n$.

Since for an AWGN channel, n_n is a complex Gaussian random variable whose real and imaginary components have variance

$$E \{ [\text{Re}(n_n)]^2 \} = E \{ [\text{Im}(n_n)]^2 \} \triangleq \sigma^2 \quad (\text{A-5})$$

then

$$\text{var} \left\{ \text{Re} \left[\sum_{n \in \eta} (\hat{x}_n - x_n)^* n_n \right] \right\} = \sigma^2 \sum_{n \in \eta} |x_n - \hat{x}_n|^2 \quad (\text{A-6})$$

and the conditional pairwise error probability $P(\underline{x} \rightarrow \hat{\underline{x}} | \underline{\phi})$ is given by

$$\begin{aligned} P(\underline{x} \rightarrow \hat{\underline{x}} | \underline{\phi}) &= \Pr \left\{ \text{Re} \left[\sum_{n \in \eta} (\hat{x}_n - x_n)^* n_n \right] \right. \\ &\quad \left. > \text{Re} \left[\sum_{n \in \eta} x_n (x_n - \hat{x}_n)^* e^{j\phi_n} \right] \right\} \\ &= Q \left(\frac{\text{Re} \left[\sum_{n \in \eta} x_n (x_n - \hat{x}_n)^* e^{j\phi_n} \right]}{\sigma \sqrt{\sum_{n \in \eta} |x_n - \hat{x}_n|^2}} \right) \end{aligned} \quad (\text{A-7})$$

where $\underline{\phi} = (\phi_1, \phi_2, \dots, \phi_N)$ is the sequence of carrier phase errors and $Q(x)$ is the Gaussian integral defined by

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp \left(-\frac{y^2}{2} \right) dy \quad (\text{A-8})$$

To simplify Eq. (A-7), proceed as follows. Since for constant envelope signals

$$2\text{Re} \{ x_n (x_n - \hat{x}_n)^* \} = |x_n - \hat{x}_n|^2 \quad (\text{A-9})$$

$$2\text{Im} \{ x_n (x_n - \hat{x}_n)^* \} = (x_n - \hat{x}_n)^* (x_n + \hat{x}_n)$$

the numerator of the argument of the Gaussian integral in Eq. (A-7) becomes

$$\begin{aligned}
& \sum_{n \in \eta} \operatorname{Re} \{x_n (x_n - \hat{x}_n)^* e^{j\phi_n}\} \\
&= \frac{1}{2} \sum_{n \in \eta} |x_n - \hat{x}_n|^2 \cos \phi_n + j(x_n - \hat{x}_n)^* (x_n + \hat{x}_n) \sin \phi_n \\
&= \frac{1}{2} \sum_{n \in \eta} |x_n - \hat{x}_n|^2 \left\{ \cos \phi_n + j \frac{(x_n + \hat{x}_n)}{(x_n - \hat{x}_n)} \sin \phi_n \right\}
\end{aligned} \tag{A-10}$$

Performing some further trigonometric simplification of Eq. (A-10) gives the desired result

$$\begin{aligned}
\sum_{n \in \eta} \operatorname{Re} \{x_n (x_n - \hat{x}_n)^* e^{j\phi_n}\} &= \sum_{n \in \eta} |x_n - \hat{x}_n| \cos(\phi_n - \eta_n) \\
\eta_n &= \tan^{-1} \alpha_n \\
\alpha_n &\triangleq \sqrt{\frac{4 - |x_n - \hat{x}_n|^2}{|x_n - \hat{x}_n|^2}} = \sqrt{\frac{4 - \delta_n^2}{\delta_n^2}}
\end{aligned} \tag{A-11}$$

The argument of the Gaussian integral in Eq. (A-7) is in the form a/\sqrt{b} . For $a > 0$, one can upper bound this integral by¹

$$Q\left(\frac{a}{\sqrt{b}}\right) \leq \frac{1}{2} e^{-a^2/2b} \tag{A-12}$$

¹Note that for perfect carrier demodulation, i.e., $\phi = 0$, one always has $a > 0$.

Since for any λ , one has $(a - 2\lambda b)^2 > 0$, rearranging this inequality gives the equivalent form

$$\frac{a^2}{b} \geq 4\lambda a - 4\lambda^2 b \tag{A-13}$$

Thus, for $a > 0$,

$$Q\left(\frac{a}{\sqrt{b}}\right) \leq \frac{1}{2} \exp \{-2\lambda[a - \lambda b]\} \tag{A-14}$$

For $a < 0$, one must use the loose upper bound

$$Q\left(\frac{a}{\sqrt{b}}\right) = Q\left(-\frac{|a|}{\sqrt{b}}\right) = 1 - Q\left(\frac{|a|}{\sqrt{b}}\right) \leq 1 \tag{A-15}$$

Finally, using Eq. (A-11) together with Eqs. (A-14) and (A-15) in Eq. (A-7) gives the desired upper bound on pairwise error probability as

$$P(\underline{x} \rightarrow \hat{\underline{x}} | \phi; \lambda) \leq \begin{cases} \frac{1}{2} \exp \left\{ -\frac{E_s}{4N_0} \sum_{n \in \eta} 4\lambda(\cos \phi_n + \alpha_n \sin \phi_n - \lambda) \delta_n^2 \right\}; \\ \sum_{n \in \eta} \delta_n^2 (\cos \phi_n + \alpha_n \sin \phi_n) > 0 \\ 1; \quad \sum_{n \in \eta} \delta_n^2 (\cos \phi_n + \alpha_n \sin \phi_n) \leq 0 \end{cases} \tag{A-16}$$

In Eq. (A-16), use has been made of the fact that for the unnormalized system, $1/2\sigma^2 = E_s/N_0$ where E_s is the symbol energy and N_0 the noise spectral density, and λ has been replaced by the normalized quantity $\lambda\sigma^2$. Also, note that if Eq. (A-16) is minimized over λ , then it is identically in the form of a Chernoff bound.

Appendix B

Derivation of the Integration Region \mathcal{R} for Suppressed Carrier Tracking

First it will be shown that for *any error event path*, the intersection of the intervals $0 \leq |\phi| \leq \pi/M$ and $0 \leq |\phi - \phi_1| \leq \pi/2$, where ϕ_1 is defined in Eq. (10), is indeed $0 \leq |\phi| \leq \pi/M$, which then defines the region \mathcal{R} for the no-interleaving case. This is equivalent to showing that for *any error event path*, $\pi/2 - \phi_1 \geq \pi/M$. From Eq. (10), this inequality can be expressed as

$$\frac{\sum_{n \in \eta} \sqrt{\delta_n^2 (4 - \delta_n^2)}}{\sum_{n \in \eta} \delta_n^2} \leq \cot \frac{\pi}{M} \quad (\text{B-1})$$

or, equivalently,

$$\sum_{n \in \eta} \sqrt{\delta_n^2 (4 - \delta_n^2)} \leq \sum_{n \in \eta} \frac{\delta_n^2}{\tan \frac{\pi}{M}} \quad (\text{B-2})$$

Equation (B-2) will be satisfied if for each $n \in \eta$,

$$\delta_n^2 (4 - \delta_n^2) \leq \frac{\delta_n^4}{\tan^2 \frac{\pi}{M}} \quad (\text{B-3})$$

or, equivalently,

$$\delta_n^2 \geq \frac{4 \tan^2 \frac{\pi}{M}}{1 + \tan^2 \frac{\pi}{M}} = 4 \sin^2 \frac{\pi}{M} \quad (\text{B-4})$$

However, for an MPSK signaling set, the *smallest* squared Euclidean distance occurs between adjacent points in the constellation and has value $4 \sin^2 \pi/M$. Thus, Eq. (B-4) is satisfied for all $n \in \eta$. Q.E.D.

For the interleaving case, it must be shown that for $0 \leq |\phi_n| \leq \pi/M$,

$$\sum_{n \in \eta} \delta_n^2 (\cos \phi_n + \alpha_n \sin \phi_n) \geq 0 \quad (\text{B-5})$$

where, from Eq. (5),

$$\alpha_n = \sqrt{\frac{4 - \delta_n^2}{\delta_n^2}} \quad (\text{B-6})$$

Since, if all the ϕ_n 's are equal to $-\pi/M$, the left-hand side of Eq. (B-5) is most negative, then, equivalently, it must be shown that

$$\sum_{n \in \eta} \delta_n^2 \left(\cos \frac{\pi}{M} - \alpha_n \sin \frac{\pi}{M} \right) \geq 0 \quad (\text{B-7})$$

The inequality in Eq. (B-7) is satisfied if each term in the sum is greater than or equal to zero. Thus, it must be shown that

$$\cos \frac{\pi}{M} - \sqrt{\frac{4 - \delta_n^2}{\delta_n^2}} \sin \frac{\pi}{M} \geq 0 \quad (\text{B-8})$$

or, equivalently,

$$\tan^2 \frac{\pi}{M} \leq \frac{\delta_n^2}{4 - \delta_n^2} \quad (\text{B-9})$$

which is the identical inequality to Eq. (B-3) whose validity was established above. Q.E.D.